Notes and Examples for Part 2:

- Functions and Domain
- Operations with Functions
- Function Composition
- Inverses

PreCalculus
Notes

Name: ______________
Date: ______________

Chapter 1.1

I. Vocabulary

a. **Relation:** any relationship that can be expressed by ordered pairs.

b. **Domain:** constraints on x values. Input.

c. **Range:** constraints on y values. Output.

d. **Function:** a relation in which each element of the domain is paired with exactly one element in the range.

e. **Vertical Line Test:** every vertical line only passes through the graph once. Proves a function

f. **Functional Notation:** An equation y in terms of x can be written $y = f(x)$

g. **Independent Variable:** x (not dependent on y)

h. **Dependent Variable:** y (dependent on x.)

II. Number Sets

a. **Naturals** - counting numbers $N = \{1, 2, 3 \ldots\}$

b. **Wholes** - natural numbers and zero $\{0, 1, 2, 3 \ldots\}$

c. **Integers** - whole numbers and opposites $\{\ldots -2, -1, 0, 1, 2 \ldots\}$

d. **Rational** - can be expressed as the ratio of 2 integers

Number Sets Continued…

e. Irrational – cannot be expressed as the ratio of 2 integers. ex. π, $\sqrt{2}$

f. Real – all rationals and irrationals.

g. Complex – numbers in the form $x+yi$ where x and y are real numbers and $i=\sqrt{-1}$

III. Examples

State the domain and range of each relation. Then state whether the relation is a function. Write yes or no. Explain using a complete sentence.

1. {(-1, 2), (3, 10), (-2, 20), (3, 11)}

domain: {-2, -1, 3}

range: {2, 10, 11, 20)

2. {(1, 4), (2, 8), (3, 24)}

domain: {1, 2, 3}

range: {4, 8, 24}

Given $f(x) = |3x - 4| + 5$, find each value.

3. $f\left(\frac{1}{3}\right) = |3 \cdot \frac{1}{3} - 4| + 5$
$= |1-4| + 5$
$= |-3| + 5$
$= 3 + 5 = \boxed{8}$

4. $f(0.5) = |3 \cdot 0.5 - 4| + 5$
$= |1.5 - 4| + 5$
$= |-2.5| + 5$
$= 2.5 + 5 = \boxed{7.5}$

5. $f(-0.5) = |3 \cdot 0.5 - 4| + 5$
$= |-1.5 - 4| + 5$
$= |-5.5| + 5$
$= 5.5 + 5 = \boxed{10.5}$

6. $f(5d) = |3 \cdot 5d - 4| + 5$
$= \boxed{|15d - 4| + 5}$

Domain is asking what will cause the function to be undefined or complex. The values of x that cause the function to be undefined or complex will not be in the domain (if you can't graph it, it's not in the domain). Therefore, domain consists of only **real** numbers!!

Finding the Domain of a Function Defined by an Equation

1. Start with the domain as the set of real numbers.
2. If the equation has a denominator, exclude any numbers that give a zero denominator.
3. If the equation has a radical of even index, exclude any numbers that cause the expression inside the radical to be negative.

Show all graphs!!

State the domain of each function.

7. $f(x) = \frac{x-2}{x+4}$

$X+4=0$

$X \neq -4$

all $\mathbb{R}$ except $X \neq -4$

8. $f(x) = \frac{1}{|2x+5|}$

$|2X+5| = 0$

$2X = -5$

$X = -5/2$

all $\mathbb{R}$ except $X \neq -\frac{5}{2}$

9. $f(x) = \frac{x-7}{x^2-1} = \frac{X-7}{(X+1)(X-1)}$

$X \neq 1, \quad X \neq -1$

10. $f(x) = \frac{x^2+25}{x^2-25} = \frac{X^2+25}{(X+5)(X-5)}$

$X \neq 5, \quad X \neq -5$

11. $f(x) = \sqrt{x^2-25}$

$X^2 - 25 \geq 0$

$X^2 \geq 25$

$X \geq 5 \quad X \leq -5$

need to flip the sign when $\sqrt{\ }$ result is a negative.

$X \leq -5 \quad X \geq 5$

graph to check...

12. $f(x) = \frac{x-10}{\sqrt{x^2-16}}$

$X^2 - 16 > 0$

$X^2 > 16$

$X > 4 \quad X < -4$

flip the sign...

$X < -4 \quad X > 4$

PreCalculus
Notes

Name: ____________________
Date: ________________

Chapter 1.2 (Day 1)

I. Operations with functions

Given the two functions $f(x)$ and $g(x)$, the following operations apply:

a. $[f+g](x) =$ f(x) + g(x) sum

b. $[f-g](x) =$ f(x) - g(x) difference

c. $[f \cdot g](x) =$ f(x) · g(x) product

d. $\left[\frac{f}{g}\right](x) =$ f(x) ÷ g(x) or f(x)/g(x), g(x) ≠ 0.

** When working with operations and functions, make sure to identify the domain of each individual function, because if there is something that is not able to be in the one function, it cannot be in the function that results when you add, subtract, multiply or divide. Then, check the domain of the new function that is created (when you add, subtract, multiply or divide).

II. Examples

1. If $f(x) = \dfrac{2}{x+4}$ and $g(x) = x^2 - 16$, find the following values:

a. $[f+g](x) = f(x) + g(x)$

$$= \frac{2}{x+4} + (x^2-16) = \frac{2}{x+4} + \frac{(x^2-16)(x+4)}{x+4} = \frac{2 + x^3 + 4x^2 - 16x - 64}{x+4}$$

$$= \boxed{\frac{x^3+4x^2-16x-62}{x+4}} \qquad x \neq -4$$

b. $[f \cdot g](x) = f(x) \cdot g(x)$

$$= \left(\frac{2}{x+4}\right)\left(\frac{x^2-16}{1}\right) = \frac{2(x^2-16)}{x+4} = \frac{2(x+4)(x-4)}{x+4}$$

$$= \boxed{2(x-4)}$$

$x \neq -4$

2. If $f(x)=\dfrac{5}{2-x}$ and $g(x)=\dfrac{4}{x-4}$, find the following values:

a. $[f-g](x) = f(x) - g(x)$

$$= \left(\frac{5}{2-x}\right) - \left(\frac{4}{x-4}\right) = \frac{5(x-4)-4(2-x)}{(2-x)(x-4)}$$

$$= \frac{5x-20-8+4x}{(2-x)(x-4)} = \boxed{\frac{9x-28}{(2-x)(x+4)}} \qquad x \neq 2,\quad x \neq -4$$

do not multiply out the denominator!

b. $\left[\dfrac{f}{g}\right](x) = f(x) \div g(x)$

$$= \frac{5}{2-x} \div \frac{4}{x-4} = \frac{5}{2-x} \cdot \frac{x-4}{4}$$

$$= \boxed{\frac{5(x-4)}{4(2-x)}} \qquad x \neq 2,\quad x \neq -4$$

Practice Problems on page 17
Mandatory: 12, 14. Suggested: 11, 13.

PreCalculus
Notes

Name: ____________________
Date: ______________

Chapter 1.2 (Day 2)

I. More Functions

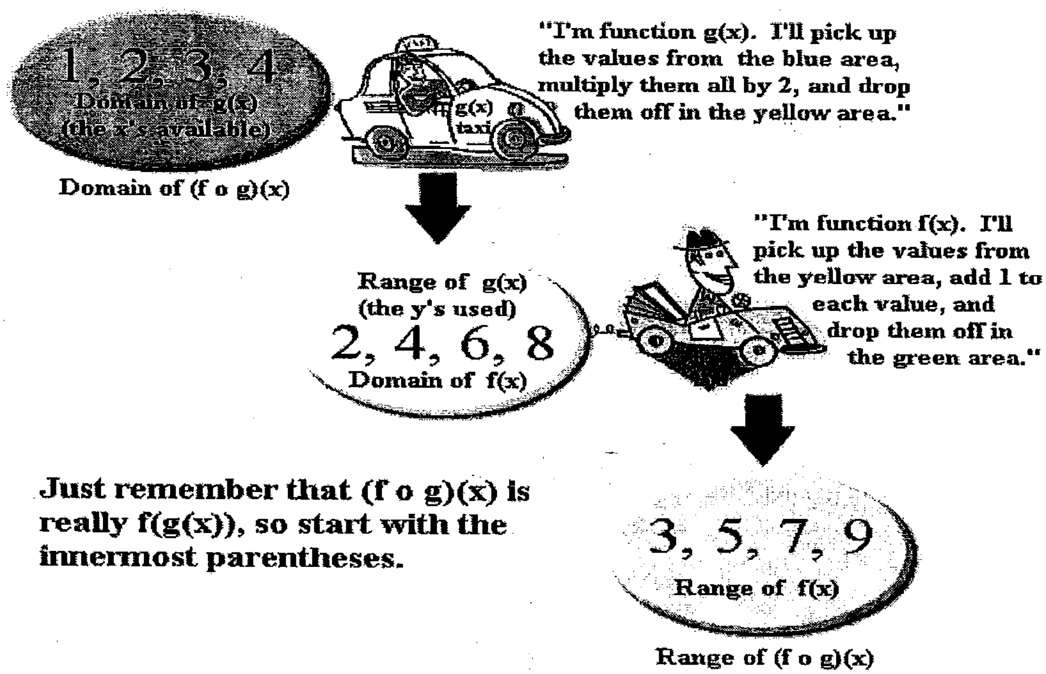
a. Functions can be combined by using function composition. In a composition, a function is performed, and then a second function is performed on the result of the first function.

i. The function formed by composing two functions f and g is called the composite function of f and g. It is denoted by $(f \circ g)(x)$ or $f(g(x))$.

b. The range of the first function becomes the domain of the second function.

c. The math definition of composition of functions is as follows: Given functions f and g, the composite function $f \circ g$ can be described by the following equation: $[f \circ g](x) = f[g(x)]$. The domain of $f \circ g$ includes all of the elements x in the domain of g for which $g(x)$ is in the domain of f.

f(x) = x + 1 g(x) = 2x
Together they create (f o g)(x).



II. Examples

1. If $f(x) = x^2 - 6$ and $g(x) = \dfrac{x}{x+1}$, find $f[g(1)]$.

$f(g(1)) =$

$f\left(\dfrac{1}{1+1}\right) = f\left(\dfrac{1}{2}\right) = \left(\dfrac{1}{2}\right)^2 - 6 = \dfrac{1}{4} - 6 = \dfrac{1}{4} - \dfrac{24}{4} = \boxed{\dfrac{-23}{4}}$

2. If $f(x) = x^2 - 6$ and $g(x) = \dfrac{x}{x+1}$, find $f[g(-1)]$

$f(g(-1)) =$

$f\left(\dfrac{-1}{-1+1}\right) = f\left(\dfrac{-1}{0}\right)$ ~ undifined!

Find $f[g(x)]$ and $g[f(x)]$. State the domain of the composition.

3. $f(x) = \dfrac{1}{2}x - 7$ and $g(x) = x + 6$.

$f(g(x)) = f(x+6) = \dfrac{1}{2}(x+6) - 7 = \dfrac{1}{2}x + 3 - 7 = \boxed{\dfrac{1}{2}x - 4}$

Domain: all reals.

$g(f(x)) = g\left(\dfrac{1}{2}x - 1\right) = \left(\dfrac{1}{2}x - 1\right) + 6 = \boxed{\dfrac{1}{2}x - 1}$

Domain: all reals

4. $f(x) = x + 1$ and $g(x) = \dfrac{x}{x+5}$.

$f(g(x)) = f\left(\dfrac{x}{x+5}\right) = \left(\dfrac{x}{x+5}\right) + 1 = \dfrac{x}{x+5} + \dfrac{x+5}{x+5}$

$= \dfrac{x + x + 5}{x+5} = \boxed{\dfrac{2x+5}{x+5}}$ Domain: $x \neq -5$

$g(f(x)) = g(x+1) = \dfrac{(x+1)}{(x+1)+5} = \boxed{\dfrac{x+1}{x+6}}$

Domain: $x \neq -6$

5. $f(x)=\dfrac{x+2}{x-1}$ and $g(x)=\dfrac{x-5}{x+4}$

$$f(g(x)) = f\left(\frac{x-5}{x+4}\right) = \frac{\left(\frac{x-5}{x+4}\right)+2}{\left(\frac{x-5}{x+4}\right)-1} = \frac{\frac{x-5}{x+4}+\frac{2(x+4)}{x+4}}{\frac{x-5}{x+4}-\left(\frac{x+4}{x+4}\right)}$$

$$= \frac{\frac{x-5+2(x+4)}{x+4}}{\frac{x-5-(x+4)}{x+4}} = \frac{\frac{x-5+2x+8}{x+4}}{\frac{x-5-x-4}{x+4}}$$

$$= \frac{\frac{3x+3}{x+4}}{\frac{-9}{x+4}} = \frac{3x+3}{x+4}\cdot\frac{x+4}{-9} = \frac{3x+3}{-9}$$

$$= \frac{3(x+1)}{-9} = \boxed{\frac{x+1}{-3}}$$

Domain: $x \neq -4$

$$g(f(x)) = g\left(\frac{x+2}{x-1}\right) = \frac{\left(\frac{x+2}{x-1}\right)-5}{\left(\frac{x+2}{x-1}\right)+4} = \frac{\frac{x+2}{x-1}-\frac{5(x-1)}{x-1}}{\frac{x+2}{x-1}+\frac{4(x-1)}{x-1}}$$

$$= \frac{\frac{x+2-5x+5}{x-1}}{\frac{x+2+4x-4}{x-1}} = \frac{\frac{-4x+7}{x-1}}{\frac{5x-2}{x-1}} = \boxed{\frac{-4x+7}{5x-2}}$$

Domain: $x \neq 1$

$x \neq \frac{2}{5}$

Over for homework problems…..

Chapter 1.2 – Function Composition Practice and Quiz Review

Find $[f \circ g](x)$ for each $f(x)$ and $g(x)$. Also state the domain of the composition.

Work on a separate sheet of paper.

1. $f(x)=\dfrac{3x+5}{2}$ and $g(x)=\dfrac{2x-5}{3}$

2. $f(x)=x^2$ and $g(x)=\dfrac{1}{x^3}$.

3. $f(x)=\dfrac{x}{x-2}$ and $g(x)=\dfrac{3}{x}$

4. $f(x)=\dfrac{x-1}{x-2}$ and $g(x)=\dfrac{x-3}{x-4}$

5. $f(x)=x^2-16$ and $g(x)=\sqrt{x}$

6. $f(x)=x^2-3x$ and $g(x)=\sqrt{x+2}$

7. $f(x)=\sqrt{x-2}$ and $g(x)=3x$

8. $f(x)=x^2-4$ and $g(x)=\sqrt{3x}$

9. $f(x)=\sqrt{3x}$ and $g(x)=x^2-4$

10. Find $(f+g)(x)$, $(f-g)(x)$, $(f \cdot g)(x)$, and $\left(\dfrac{f}{g}\right)(x)$, given the following:

    $f(x)=\dfrac{1}{x}$ and $g(x)=7-x$

11. Find $(f+g)(x)$, $(f-g)(x)$, $(f \cdot g)(x)$, and $\left(\dfrac{f}{g}\right)(x)$, given the following:

    $f(x)=\dfrac{1}{2-3x}$ and $g(x)=\dfrac{2}{3x-2}$

12. Find the domain of the following functions:

    (a) $f(x)=\dfrac{x+4}{x^2-2x-3}$ (b) $g(x)=\dfrac{1}{\sqrt{x^2-64}}$ (c) $h(x)=\sqrt{3-2x}$

PreCalculus
Notes

Name: ______________

Date: ___________

Functions with Inverses

A function *f* has an inverse if and only if when its graph is reflected about the line $y = x$, the result is the graph of a function (passes the vertical line test).

But this can be simplified. We can tell before we reflect the graph whether or not any vertical line will intersect more than once by looking at how horizontal lines intersect the original graph!

Horizontal Line Test

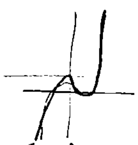
Let *f* be a function.

- If any horizontal line intersects the graph of *f* more than once, then *f* does not have an inverse.
- If no horizontal line intersects the graph of *f* more than once, then *f* does have an inverse.

Examples - Determine if the function has an inverse by graphing.

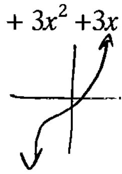
1. $f(x) = x^2(x - 3)$

No.



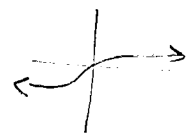
2. $f(x) = x^3 + 3x^2 + 3x$

Yes.



3. $f(x) = \sqrt[3]{x}$

yes



Steps for finding the inverse of a function *f*.

1. Replace $f(x)$ by y in the equation describing the function.
2. Interchange x and y. In other words, replace every x by a y and vice versa.
3. Solve for y.
4. Replace y by $f^{-1}(x)$.

Example 1: The equation for Group 1's set of points is $f(x) = 3x + 6$. Find the equation of the inverse.

$$y = 3x + 6$$

$$x = 3y + 6$$
$$-6 \qquad -6$$

$$\frac{x-6}{3} = \frac{3y}{3}$$

$$\frac{x}{3} - 2 = y$$

$$\boxed{\frac{x}{3} - 2 = f^{-1}(x)}$$

Example 2: Find the equation of the inverse of $f(x) = -2x + 3$.

$$y = -2x + 3$$
$$x = -2y + 3$$
$$-3 \qquad -3$$
$$x - 3 = -2y$$
$$-\frac{x}{2} + \frac{3}{2} = y$$

$$\boxed{-\frac{x}{2} + \frac{3}{2} = f^{-1}(x)}$$

Example 3: Find the equation of the inverse of $f(x) = 2x^3 - 5$.

$$y = 2x^3 - 5$$
$$x = 2y^3 - 5$$
$$x + 5 = 2y^3$$
$$\frac{x}{2} + \frac{5}{2} = y^3$$

$$\sqrt[3]{\frac{x+5}{2}} = y$$

$$\boxed{\sqrt[3]{\frac{x+5}{2}} = f^{-1}(x)}$$

Example 4: Find the equation of the inverse of $f(x) = 2(x + 4)^7$

$$y = 2(x+4)^7$$
$$x = 2(y+4)^7$$
$$\frac{x}{2} = (y+4)^7$$
$$\sqrt[7]{\frac{x}{2}} = y + 4$$
$$\sqrt[7]{\frac{x}{2}} - 4 = y$$

$$\boxed{\sqrt[7]{\frac{x}{2}} - 4 = f^{-1}(x)}$$

Homework: Finding Inverses Worksheet

PreCalculus
Notes

Name: ____________________
Date: ________________

Inverses and Function Composition

Let's explore the relationship between a function and it's inverse using function composition and your homework problems.

1. $f(x) = x^3$ and $g(x) = \sqrt[3]{x}$

   You found that these were inverses in your homework. Now, find $f(g(x))$.

   $f(g(x)) = f(\sqrt[3]{x}) = (\sqrt[3]{x})^3 = \boxed{x}$

   Find $g(f(x))$.

   $g(f(x)) = g(x^3) = \sqrt[3]{x^3} = \boxed{x}$

   What is the result of both compositions? $\boxed{x}$

2. $h(x) = \dfrac{1}{x}$ and $i(x) = \dfrac{1}{x}$

   Perform the function compositions $h(i(x))$ and $i(h(x))$. What are the results?

   $h(i(x)) = h\left(\frac{1}{x}\right) = \dfrac{1}{\frac{1}{x}} = 1 \div \frac{1}{x} = 1 \cdot \frac{x}{1} = \boxed{x}$

   $i(h(x)) = i\left(\frac{1}{x}\right) = \dfrac{1}{\frac{1}{x}} = \boxed{x}$

3. $w(x) = 2x+1$ and $z(x) = \dfrac{x-1}{2}$

   $w(z(x)) = w\left(\frac{x-1}{2}\right) = 2\left(\frac{x-1}{2}\right) + 1 = x - 1 + 1 = \boxed{x}$

   $z(w(x)) = z(2x+1) = \dfrac{(2x+1)-1}{2} = \dfrac{2x}{2} = \boxed{x}$

We have found that we can show that two functions are inverses if

$f(g(x)) = g(f(x)) = x$.

More Practice: Confirm that the functions are inverses.

4. $g(x) = x^2 + 1$ and $h(x) = \sqrt{x-1}$

$$g(h(x)) = g(\sqrt{x-1}) = (\sqrt{x-1})^2 + 1 = x - 1 + 1 = \boxed{x}$$

$$h(g(x)) = h(x^2+1) = \sqrt{(x^2+1)-1} = \sqrt{x^2} = \boxed{x}$$

Yes - inverses confirmed.

5. $r(x) = \sqrt[5]{2x+1}$ and $s(x) = \dfrac{x^5 - 1}{2}$

$$r(s(x)) = r\left(\frac{x^5-1}{2}\right) = \sqrt[5]{2\left(\frac{x^5-1}{2}\right)+1} = \sqrt[5]{x^5-1+1} = \sqrt[5]{x^5} = \boxed{x}$$

$$s(r(x)) = s(\sqrt[5]{2x+1}) = \frac{(\sqrt[5]{2x+1})^5 - 1}{2} = \frac{2x+1-1}{2} = \frac{2x}{2} = \boxed{x}$$

Yes - inverses confirmed.

6. $f(x) = \dfrac{x+1}{x-1}$ and $g(x) = \dfrac{-(x+1)}{1-x}$

$$f(g(x)) = f\left(\frac{-(x+1)}{1-x}\right) = \frac{\left(\frac{-(x+1)}{1-x}\right)+1}{\left(\frac{-(x+1)}{1-x}\right)-1} = \frac{\frac{-(x+1)}{1-x} + \frac{1-x}{1-x}}{\frac{-(x+1)}{1-x} - \frac{1-x}{1-x}}$$

$$= \frac{\frac{-x-1+1-x}{1-x}}{\frac{-x-1-1+x}{1-x}} = \frac{\frac{-2x}{1-x}}{\frac{-2}{1-x}} = \frac{-2x}{1-x} \cdot \frac{1-x}{-2} = \boxed{x}$$

$$g(f(x)) = g\left(\frac{x+1}{x-1}\right) = \frac{-\left(\left(\frac{x+1}{x-1}\right)+1\right)}{1-\left(\frac{x+1}{x-1}\right)} = \frac{-\left(\frac{x+1+x-1}{x-1}\right)}{\frac{x-1+(-x-1)}{x-1}} = \frac{\left(\frac{-2x}{x-1}\right)}{\frac{-2}{x-1}}$$

$$= \frac{-2x}{x-1} \cdot \frac{x-1}{-2} = \boxed{x}$$

Yes - inverses confirmed!

Homework: Showing Inverses by Composition Worksheet